Short Distance Analysis in Algebraic Quantum Field Theory*

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Abstract

Within the framework of algebraic quantum field theory a general method is presented which allows one to compute and classify the short distance (scaling) limit of any algebra of local observables. The results can be used to determine the particle and symmetry content of a theory at very small scales and thereby give an intrinsic meaning to notions such as "parton" and "confinement". The method has been tested in models.

1 Introduction

Within the Lagrangean approach to quantum field theory, a powerful tool for the analysis of the ultraviolet properties of models is based on scaling (renormalization group) transformations. They allow one to interpret the theory at small space-time scales and have led to fundamental concepts such as the notion of parton, confinement, asymptotic freedom etc.

This approach has been substantial for the present theoretical understanding of high energy physics. Nevertheless, it is not completely satisfactory since it is based on the (gauge) fields appearing in the Lagrangean which in general do not admit a direct physical interpretation. It therefore seems desirable to establish a framework for the short distance analysis which relies on observables only. Since observables are typically composed of several elementary fields this may not seem an easy task if one thinks e.g. of a generalization of renormalization group equations. But it turns out that there is a conceptually simple solution of this problem [1] in the general algebraic framework of local quantum physics [2].

In [1] we have established a method which allows one to define scaling transformations and the scaling limit for any given algebra of local observables $\mathcal{A}$. For the convenience of the reader who is not familiar with the algebraic setting, we briefly list the relevant assumptions.

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1. (Locality) We suppose that the local observables of the underlying theory generate a net of local algebras over d-dimensional Minkowski space $\mathbb{R}^d$, i.e. an inclusion preserving map

$$\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$$

from the set of open, bounded regions $\mathcal{O}$ in Minkowski space to unital C*-algebras $\mathcal{A}(\mathcal{O})$. The algebra generated by all local algebras $\mathcal{A}(\mathcal{O})$ (as a C*-inductive limit) will be denoted by $\mathcal{A}$. The net is supposed to satisfy the principle of locality (Einstein causality), i.e. operators localized in spacelike separated regions commute.

2. (Covariance) The Poincaré group $\mathcal{P}_+^\uparrow$ is represented by automorphisms of the net. Thus for each $(\Lambda, x) \in \mathcal{P}_+^\uparrow$ there is an $\alpha_{\Lambda,x} \in \mathrm{Aut}\mathcal{A}$ such that, in an obvious notation,

$$\alpha_{\Lambda,x}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O} + x)$$

for any region $\mathcal{O}$. We amend this fundamental postulate by a continuity condition and assume that for any $A \in \mathcal{A}$ the function $(\Lambda, x) \rightarrow \alpha_{\Lambda,x}(A)$ is strongly continuous.

3. (States) Physical states are described by positive, linear and normalized functionals ω on $\mathcal{A}$. By the GNS-construction, any state ω gives rise to a representation π_ω of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$, and there exists a vector $\Omega_\omega \in \mathcal{H}_\omega$ such that

$$\omega(A) \;=\; (\Omega_\omega, \pi_\omega(A)\Omega_\omega),\; A \in \mathcal{A}.$$

The state describing the vacuum is distinguished by the fact that, on the corresponding Hilbert space $\mathcal{H}_\omega$, there is a continuous unitary representation $U_\omega(\Lambda, x)$ of the Poincaré group $\mathcal{P}_+^\uparrow$ which leaves the vector Ω_ω invariant, satisfies the relativistic spectrum condition (positivity of energy) and implements the action of $\mathcal{P}_+^\uparrow$ on the observables,

$$U_\omega(\Lambda, x)\pi_\omega(A)U_\omega(\Lambda, x)^{-1} \;=\; \pi_\omega(\alpha_{\Lambda,x}(A)),\; A \in \mathcal{A}.$$

Any state of physical interest is assumed to be locally normal to the vacuum state.

2 Scaling algebra and scaling limit

What is required in order to carry over the ideas of renormalization group analysis to this algebraic setting? One first has to proceed from the given net and automorphisms $\mathcal{A}, \alpha$ at spacetime scale $\lambda = 1$ (in appropriate units) to the corresponding nets $\mathcal{A}^{(\lambda)}, \alpha^{(\lambda)}$ describing the theory at arbitrary scale $\lambda \in \mathbb{R}_+$. This is easily accomplished by setting for given λ

$$\mathcal{A}^{(\lambda)}(\mathcal{O}) \doteq \mathcal{A}(\lambda\mathcal{O}), \quad \alpha^{(\lambda)}_{\Lambda,x} \doteq \alpha_{\Lambda,\lambda x}.$$

Note that the latter nets are in general not isomorphic to the original one. They are to be regarded as different theories with scaled mass spectrum, running coupling constants etc.

In addition to this passage from the given theory to the corresponding theories at arbitrary scale one needs a way of comparing the respective observables. This is accomplished by considering functions $\underline{A}$ of the scaling parameter with values in the algebra of observables $\mathcal{A}$. For given λ the value $\underline{A}_\lambda$ of $\underline{A}$ is to be regarded as

an observable in the theory $\mathcal{A}^{(\lambda)}, \alpha^{(\lambda)}$. The graph of the function $\underline{A}$ thus provides the desired identification of observables at different scales. With this idea in mind one is led to the concept of scaling algebra.

The *scaling algebra* $\underline{\mathcal{A}}$ associated with any given local net $\mathcal{A}$ consists of functions $\underline{A} : \mathbb{R}_+ \to \mathcal{A}$. The algebraic operations in $\underline{\mathcal{A}}$ are pointwise defined by the corresponding operations in $\mathcal{A}$, and there is a C*-norm on $\underline{\mathcal{A}}$ given by

$$||\underline{A}|| = \sup_\lambda ||\underline{A}_\lambda||.$$

The local structure of $\mathcal{A}$ is lifted to $\underline{\mathcal{A}}$ by setting

$$\underline{\mathcal{A}}(\mathcal{O}) = \{\underline{A} : \underline{A}_\lambda \in \mathcal{A}^{(\lambda)}(\mathcal{O}),\ \lambda \in \mathbb{R}_+\}.$$

Hence $\mathcal{O} \to \underline{\mathcal{A}}(\mathcal{O})$ defines a net over Minkowski space and $\underline{\mathcal{A}}$ is defined as the C*-inductive limit of the local algebras $\underline{\mathcal{A}}(\mathcal{O})$. It is easily verified that this net is local. Moreover, the action of the Poincaré group in the underlying theory can be lifted to an action of automorphisms $\underline{\alpha}_{\Lambda,x}$ on $\underline{\mathcal{A}}$ which is given by

$$(\underline{\alpha}_{\Lambda,x}(\underline{A}))_\lambda \doteq \alpha^{(\lambda)}_{\Lambda,x}(\underline{A}_\lambda).$$

It is assumed that $\underline{\mathcal{A}}$ consists only of elements on which these automorphisms act strongly continuously, i.e.

$$||\,\underline{\alpha}_{\Lambda,x}(\underline{A}) - \underline{A}\,|| \to 0 \quad \text{as} \quad (\Lambda, x) \to (1, 0).$$

Heuristically speaking, the latter constraint amounts to the condition [1] that the operators $\underline{A}_\lambda$ in the graph of a given element $\underline{A} \in \underline{\mathcal{A}}$ occupy, for all values of λ, a fixed volume of "phase space". Hence, whereas the scale of spacetime changes along the graph, the scale $\hbar$ of action is kept fixed.

The structure of the physical states ω in the underlying theory at small spacetime scales can now be analyzed with the help of the scaling algebra as follows. Given ω, one defines the lift of this state to the scaling algebra at scale $\lambda \in \mathbb{R}_+$ by setting

$$\underline{\omega}_\lambda(\underline{A}) \doteq \omega(\underline{A}_\lambda), \quad \underline{A} \in \underline{\mathcal{A}}.$$

Let $\underline{\pi}_\lambda$ be the GNS–representation of $\underline{\mathcal{A}}$ which is induced by $\underline{\omega}_\lambda$. We then consider the net

$$\mathcal{O} \to \underline{\mathcal{A}}(\mathcal{O})/\mathrm{ker}\underline{\pi}_\lambda, \quad \underline{\alpha}^{(\lambda)},$$

where ker means "kernel" and $\underline{\alpha}^{(\lambda)}$ is the induced action of the Poincaré transformations $\underline{\alpha}$ on this quotient. It is important to note [1] that this net is isomorphic to the underlying theory $\mathcal{A}^{(\lambda)}, \alpha^{(\lambda)}$ at scale λ. This insight leads to the following canonical definition of the scaling limit of the theory: One first considers the limit(s) of the net of states $\{\underline{\omega}_\lambda\}_{\lambda \searrow 0}$. By standard compactness arguments, this net has always a non–empty set $\{\underline{\omega}_0\}$ of limit points. The following facts about these limit states have been established in [1]:

1. The set $\{\underline{\omega}_0\}$ does not depend on the chosen physical state ω.
2. Each $\underline{\omega}_0$ is a vacuum state on $\underline{\mathcal{A}}$ which is pure in $d > 2$ spacetime dimensions.

Denoting the GNS–representation corresponding to given $\underline{\omega}_0$ by $(\underline{\pi}_0, \underline{\mathcal{H}}_0)$ one can then define in complete analogy to the case $\lambda > 0$ the net

$$\mathcal{O} \to \mathcal{A}^{(0)} \doteq \underline{\mathcal{A}}(\mathcal{O})/\ker \underline{\pi}_0, \quad \alpha^{(0)} \doteq \underline{\alpha}^{(0)}$$

and the corresponding vacuum state $\omega_0 \doteq \mathrm{proj}\, \underline{\omega}_0$, where proj denotes the projection of the respective state to the quotient algebra. This net is to be interpreted as *scaling limit of the underlying theory.* It should be remarked that in view of the possible appearance of several limit points $\underline{\omega}_0$ this limit may not be unique.

The preceding steps which have led us from the original net to its scaling limit(s) are pictorially described in the diagram

$$\mathcal{A},\, \alpha \longrightarrow \underline{\mathcal{A}},\, \underline{\alpha} \longrightarrow \{\mathcal{A}^{(0)},\, \alpha^{(0)}\}.$$

3 Classification of theories

The scaling limits of local nets of observable algebras can be classified according to the following three mutually exclusive general possibilities.

Classification: Let $\mathcal{A}, \alpha$ be a net with properties specified in the Introduction. There are the following possibilities for the structure of the corresponding scaling limit theory.

1. The nets $\{\mathcal{A}^{(0)},\, \alpha^{(0)}\}$ are isomorphic to the trivial net $\{\mathbb{C} \cdot 1, \mathrm{id}\}$ (classical scaling limit)
2. The nets $\{\mathcal{A}^{(0)},\, \alpha^{(0)}\}$ are isomorphic and non-trivial (quantum scaling limit)
3. Not all of the nets $\{\mathcal{A}^{(0)},\, \alpha^{(0)}\}$ are isomorphic (degenerate scaling limit)

In theories with a classical scaling limit all correlations between quantum observables in $\underline{\mathcal{A}}$ disappear at small scales, hence the terminology. Recently, examples of local nets with a classical scaling limit have been constructed in [3]. They satisfy the standard conditions for nets of physical interest, such as weak additivity, wedge-duality, nuclearity etc. But in contrast to nets generated by Wightman fields, they contain only operators which exhibit a very singular (non–temperate) short distance behavior. To some extent these examples mimic the ultraviolet problems which one expects to meet in non–renormalizable theories or theories without ultraviolet fixed point. It is of interest that theories with a classical scaling limit can be characterized by their phase space properties in terms of nuclearity criteria [4].

The case of theories with a quantum scaling limit is expected to be the generic one. Here the terminology derives from the fact that the algebras $\mathcal{A}^{(0)}$ are necessarily non–abelian if they are non–trivial [4]. Simple examples in this class are free field theories in $d = 3$ and 4 dimensions as well as some exactly solvable models for $d = 2$ [5]. It seems likely that quantum field theories with a stable ultraviolet fixed point, e.g. asymptotically free theories, also belong to this class. A clarification of this point would be very desirable.

In theories with a degenerate scaling limit it is not possible to describe the short distance properties in terms of a single scaling limit theory, the structure continually varies as λ approaches 0. Candidates for this type of behavior are theories with a large number of local degrees of freedom which strongly violate

the above mentioned nuclearity criteria. Yet these results are not yet complete. It would be of great interest, both from a conceptual and technical point of view, to clarify the relation between the phase space properties of the underlying theory and the nature of its scaling limit.

Since theories with a quantum scaling limit are of particular interest let us mention the following two general facts which have been established in [1]:

1. In any theory with a quantum scaling limit the limit net $\mathcal{A}^{(0)}$, $\alpha^{(0)}$ is dilatation invariant.

2. If this limit net satisfies the Haag–Swieca compactness criterion (which is the case whenever the underlying theory complies with a quantitative version of this criterion [4]) then the scaling limit of $\mathcal{A}^{(0)}$, $\alpha^{(0)}$ is isomorphic to itself, i.e. it is a fixed point under the above procedure.

4 Ultraparticles and ultrasymmetries

The fact that the scaling limit theories $\mathcal{A}^{(0)}$, $\alpha^{(0)}$ have all features of a local net of observable algebras allows one to introduce standard concepts for their physical interpretation. We call the particles appearing in the scaling limit theory (in the sense of Wigner's particle concept) *ultraparticles* and the global gauge symmetries of the scaling limit theory *ultrasymmetries*.

In order to determine the particle and symmetry content in the scaling limit from the net $\mathcal{A}^{(0)}$, $\alpha^{(0)}$ one has to apply the Doplicher–Roberts reconstruction theorem. By this method one can recover the physical Hilbert space of all states carrying a localizable charge, the algebra of charge carrying fields and the global gauge group [6]. The necessary prerequisite for this construction, Haag–duality, is given in the scaling limit whenever the underlying theory complies with the special condition of duality invented by Bisognano and Wichmann [1].

Of particular interest is the comparison of the particle and symmetry content of the underlying theory and of its scaling limit. There may be particles which disappear in the scaling limit (think of "hadrons"), particles which survive ("leptons") and particles which only come into existence at small scales ("quarks", "gluons").

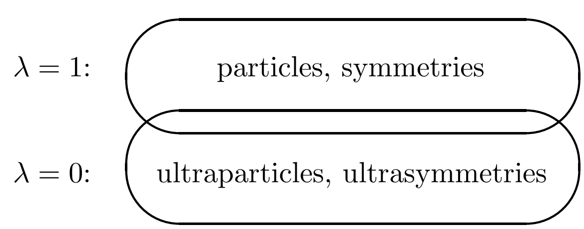

Fig.: Comparison of particle and symmetry content

In this way, notions such as parton, colour symmetry, confinement etc. acquire an unambiguous and intrinsic meaning [7].

5 Illustrations

As was already mentioned, our general method has been applied to several models [5, 7]. A nice illustration of the scenario outlined in the preceding section is provided by the Schwinger model (massless QED in $d = 2$ dimensions). It is known from the work of Lowenstein and Swieca that the net of observables in this theory is isomorphic to the net generated by a free scalar field ϕ of mass $m > 0$. The fact that the latter net has only a single (vacuum) sector is then interpreted as confinement (respectively screening) of the electric charge. It is therefore of interest to determine the ultraparticle and ultrasymmetry content of this model.

To this end one has to proceed to the associated scaling algebra $\underline{\mathcal{A}}$. A typical element of this algebra is given by

$$\lambda \to \underline{A}_\lambda = \int dx\, g(x)\, e^{i\int dy\, f(y-x)\, Z_\lambda\, \phi(\lambda y)},$$

where f, g are real test functions and Z_λ is a renormalization factor whose dependence on λ is completely arbitrary. Any such operator function is an element of the scaling algebra $\underline{\mathcal{A}}$. For the scaling limits of these operators in physical states one obtains the following results, depending on the behavior of Z_λ for $\lambda \searrow 0$:

$$\lim_\lambda \underline{A}_\lambda = \begin{cases} A_0 & : \quad Z_\lambda \simeq 1 \\ C & : \quad Z_\lambda \simeq |\ln\lambda|^{-1/2} \\ c\,1 & : \quad \text{otherwise} \end{cases} .$$

Here A_0 is the operator which one obtains if one replaces $Z_\lambda\, \phi(\lambda y)$ in the expression for $\underline{A}_\lambda$ by the massless free field $\phi_0(y)$, C is some random variable which commutes with all operators in the scaling limit theory and c is some complex number. Hence the scaling limit theory is a central extension of the net generated by the free massless field.

This result shows that one does not need to know from the outset the (anomalous) dimension of the field ϕ in order to get a well defined limit. The theory takes care of that by itself in the sense that only those elements of the scaling algebra have a non–trivial scaling limit where Z_λ has the appropriate asymptotic behavior.

In a final step one has to compute the ultrasymmetries and ultraparticles in the scaling limit theory. It turns out that, in contrast to the theory at scale $\lambda = 1$, there appear superselection sectors in the scaling limit which carry a charge of electric type (in the sense that Gauss' law holds for it) and have a particle interpretation [7]. Hence in this sense the model has a non–trivial parton structure.

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